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A remark on a paper by Alessandrini and Vessella[☆]

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Abstract

We prove that the Lipschitz constant of the Lipschitz stability result for the inverse conductivity problem proved in [G. Alessandrini, S. Vessella, Lipschitz stability for the inverse conductivity problem, *Adv. in Appl. Math.* 35 (2005) 207–241], behaves exponentially with respect to the number N of regions considered.

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Let $\Omega = B_1(0) \subset \mathbb{R}^n$, where $n \geq 2$ denotes the space dimension. Let $D = [-1/2, 1/2]^n$ be the cube of side 1 centred at the origin. We have that D is compactly contained inside Ω . Let us consider the class of admissible conductivities

$$\mathcal{A} = \left\{ \gamma \in L^\infty(\Omega) : 1/2 < \gamma < 3/2 \text{ a.e. in } \Omega \text{ and } \gamma = 1 \text{ a.e. in } \Omega \setminus D \right\}.$$

For any $\gamma \in \mathcal{A}$, we set the *Dirichlet-to-Neumann map* associated to γ as the operator $A_\gamma : H^{1/2}(\partial\Omega) \mapsto H^{-1/2}(\partial\Omega)$ given by

$$H^{1/2}(\partial\Omega) \ni \varphi \mapsto \gamma \nabla u \cdot \nu|_{\partial\Omega} \in H^{-1/2}(\partial\Omega),$$

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where $u \in H^1(\Omega)$ solves the elliptic Dirichlet problem

$$\begin{cases} \operatorname{div}(\gamma \nabla u) = 0 & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega. \end{cases}$$

Let us fix a positive integer N and let N_1 be the smallest integer such that $N \leq N_1^n$. We divide each side of the cube D into N_1 equal parts of length $h = 1/N_1$ and we let \mathcal{S}_{N_1} be the set of all the open cubes of the type $D' = (-1/2 + (j'_1 - 1)h, -1/2 + j'_1 h) \times \cdots \times (-1/2 + (j'_n - 1)h, -1/2 + j'_n h)$, where j'_1, \dots, j'_n are integers belonging to $\{1, \dots, N_1\}$. We order such cubes as follows. For any two different D' and D'' belonging to \mathcal{S}_{N_1} we say that $D < D''$ if and only if there exists $i_0 \in \{1, \dots, n\}$ such that $j'_i = j''_i$ for any $i < i_0$ and $j'_{i_0} < j''_{i_0}$.

Let D_j , $j = 1, \dots, N$, be the first N cubes, with respect to the order described above, of the set \mathcal{S}_{N_1} and let $D_0 = \Omega \setminus \bigcup_{j=1}^N \overline{D_j}$. We consider the following set of admissible conductivities

$$\mathcal{A}_N = \left\{ \gamma \in \mathcal{A}: \gamma(x) = \sum_{j=1}^N \gamma_j \chi_{D_j}(x) + \chi_{D_0}(x) \right\},$$

where χ denotes the characteristic function and γ_j , $j = 1, \dots, N$, are not prescribed constants belonging to $[1/2, 3/2]$.

It is not difficult to show that, for suitable constants A, r_0, L, M, α and λ , depending at most on n and N , the hypotheses of Theorem 7 of [1] are satisfied, therefore there exists a constant C_N , depending on n and N only, such that for any $\gamma^{(1)}, \gamma^{(2)}$ belonging to \mathcal{A}_N we have

$$\|\gamma^{(1)} - \gamma^{(2)}\|_{L^\infty(\Omega)} \leq C_N \|A_{\gamma^{(1)}} - A_{\gamma^{(2)}}\|_{\mathcal{L}(H^{1/2}(\partial\Omega), H^{-1/2}(\partial\Omega))}.$$

Our aim is to estimate from below the Lipschitz constant C_N in terms of N . In the sequel we shall always omit the dependence of the constants from the space dimension n . We have the following result, essentially based on arguments developed in [2].

Theorem. *There exist $N_0 \in \mathbb{N}$ and a positive constant K_1 such that for any $N \geq N_0$ we have*

$$C_N \geq \frac{1}{8} \exp(K_1 N^{1/(2n-1)}).$$

Proof. We define the following metric spaces. For any $N \in \mathbb{N}$, we consider (\mathcal{A}_N, d_0) where d_0 is the distance given by the $L^\infty(\Omega)$ norm. Let (\mathcal{B}, d_1) be the metric space where $\mathcal{B} = \{A_\gamma: \gamma \in \mathcal{A}\}$ and d_1 is the distance induced by the norm in $\mathcal{L}(H^{1/2}(\partial\Omega), H^{-1/2}(\partial\Omega))$. We let $\gamma^{(0)} = \chi_\Omega$.

First, we prove that for any δ , $0 < \delta \leq 1/2$, and any $N \in \mathbb{N}$ there exists $\tilde{\mathcal{A}}_N \subset \mathcal{A}_N$ such that $d_0(\tilde{\gamma}, \gamma^{(0)}) \leq \delta$ for any $\tilde{\gamma} \in \tilde{\mathcal{A}}_N$, for any two distinct points $\gamma^{(1)}, \gamma^{(2)}$ in $\tilde{\mathcal{A}}_N$ we have $d_0(\gamma^{(1)}, \gamma^{(2)}) \geq \delta$ and $\tilde{\mathcal{A}}_N$ has 3^N elements.

In fact, it is enough to take as $\tilde{\mathcal{A}}_N$ the set of functions $\tilde{\gamma}$ which assume, on each D_j , $j = 1, \dots, N$, a value among $1, 1 - \delta$ and $1 + \delta$.

We recall the following definition. For a given positive ε , $\tilde{\mathcal{B}} \subset \mathcal{B}$ is said to be an ε -net for \mathcal{B} if for every $\Lambda \in \mathcal{B}$ there exists $\tilde{\Lambda} \in \tilde{\mathcal{B}}$ such that $d_1(\Lambda, \tilde{\Lambda}) \leq \varepsilon$.

By using the arguments of the proof of Proposition 3.2 in [2], it is possible to show that we can apply Lemma 2.3 in [2] to \mathcal{B} . Therefore we have that for any ε , $0 < \varepsilon < 1/e$, there exists an ε -net for \mathcal{B} with at most $\exp(K_2(-\log \varepsilon)^{2n-1})$ elements, K_2 being a positive absolute constant.

For any $0 < \varepsilon < 1/e$ and any $N \in \mathbb{N}$, let $Q(\varepsilon, N) = \exp(K_2(-\log \varepsilon)^{2n-1})$. Let us remark that $3^N > Q(\varepsilon, N)$ if and only if $\varepsilon > \exp(-K_1 N^{1/(2n-1)}) = \varepsilon_0(N)/2$, where K_1 is a positive absolute constant. There exists $N_0 \in \mathbb{N}$ such that for any $N \geq N_0$ we have $\varepsilon_0(N) < 1/e$. Thus, for any $N \geq N_0$, if we take $\varepsilon = \varepsilon_0(N)$, we have $3^N > Q(\varepsilon, N)$, then for any δ , $0 < \delta \leq 1/2$, there exist $\gamma^{(1)}$ and $\gamma^{(2)}$ belonging to \mathcal{A}_N such that $d_0(\gamma^{(i)}, \gamma^{(0)}) \leq \delta$ for any $i = 1, 2$ and

$$\delta \leq d_0(\gamma^{(1)}, \gamma^{(2)}) \leq C_N d_1(\Lambda_{\gamma^{(1)}}, \Lambda_{\gamma^{(2)}}) \leq 2C_N \varepsilon_0(N).$$

Choosing $\delta = 1/2$, we can conclude that $C_N \geq \frac{1}{8} \exp(K_1 N^{1/(2n-1)})$. \square

References

- [1] G. Alessandrini, S. Vessella, Lipschitz stability for the inverse conductivity problem, *Adv. in Appl. Math.* 35 (2005) 207–241
- [2] M. Di Cristo, L. Rondi, Examples of exponential instability for inverse inclusion and scattering problems, *Inverse Problems* 19 (2003) 685–701.